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On the Upper Critical Dimensions of Random Spin Systems

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A set of critical exponent inequalities is proved for a large class of classical random spin systems. The inequalities imply rigorous (and probably the optimal) lower bounds for the upper critical dimensions, i.e., $d_u \geqslant 4$ for regular and random ferromagnets, $d_u \geqslant 6$ for spin glasses and random field systems.

KEY WORDS: Random spin systems; critical exponent inequalities; upper critical dimensions.

1. INTRODUCTION

In spite of considerable interest, many aspects of the random spin systems remain to be understood both physically and mathematically. In this paper I extend a simple argument used in the percolation problem[1] to a large class of random spin systems, and derive rather strong information about the possible critical phenomena. More precisely, I prove simple correlation inequalities and critical exponent inequalities (for some finite-size scaling critical exponents) for a general random spin system with short-range interactions and N-component classical spin variables. In sufficiently low dimensions these inequalities will turn out to be *inconsistent with the mean field (or canonical) scaling behavior*. Therefore, they imply that the perfect mean-field-type critical phenomena cannot take place in these dimensions. It is believed that in dimensions larger than the upper critical dimension d_u, the critical phenomena are described by mean field theory. Thus, the present inequalities imply lower bounds for the upper critical dimensions of the random spin systems. The resulting bounds ($d_u \geqslant 4$ for regular and

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random ferromagnets, $d_u \geqslant 6$ for spin glasses and random field systems) turn out to be optimal when compared with the general belief.

The method may be regarded as a standard dimensional analysis which starts, however, from rigorous correlation inequalities. Therefore, it is crucial to determine what we mean by the mean field (or canonical) scaling behavior of a given random spin system. In principle, the mean field scaling behavior may always be determined by investigating the corresponding Gaussian model,[3] so there seems to be no controversy.[4]

Being quite elementary and simple, the present method may be applied to many random (and nonrandom) statistical systems.[5] One of the interesting problems is to obtain optimal critical exponent inequalities for the Anderson localization problem and determine its upper critical dimension.

2. BASIC INEQUALITY

We consider a general N-component classical spin system whose thermal expectation in a finite lattice $\Lambda \subset Z^d$ is given by

$$\langle \cdots \rangle = Z_\Lambda^{-1} \int \prod_x d\nu(\mathbf{S}_x)(\cdots)\, e^{-\beta H} \tag{1}$$

Each spin variable $\mathbf{S}_x$ takes its values in R^N. The single-site measure $d\nu(\cdot)$ is an arbitrary, bounded measure. The most standard choice $d\nu(\mathbf{S}) = \delta(|\mathbf{S}|-1)\, d^N\mathbf{S}$ describes the Ising, XY, and Heisenberg models for $N = 1, 2$, and 3, respectively. The Hamiltonian H is

$$H = -\sum_{\langle x, y\rangle} J_{xy} \mathbf{S}_x \cdot \mathbf{S}_y - \sum_x \mathbf{H}_x \cdot \mathbf{S}_x \tag{2}$$

where $\langle x, y\rangle$ denotes the nearest neighbor pair. The exchange interactions $\{J_{xy}\}$ and the external (vector) fields $\{\mathbf{H}_x\}$ take arbitrary real values. We denote an infinite-volume limit $\Lambda \to Z^d$ of $\langle \cdots \rangle_\Lambda$ by $\langle \cdots \rangle$.

For a fixed site x, let $\Lambda(L, x)$ be a finite sublattice of Z^d which consists of the sites y with[6] $|y-x| < L$. By $\langle \cdots \rangle_{\Lambda(L,x),\text{b.c.}}$ we denote the thermal

[3] In general, the Gaussian model is obtained by replacing the single-site measure $d\nu(\mathbf{S})$ by $\text{const} \cdot \exp(-\mathbf{S}^2/2c)\, d^N\mathbf{S}$. The Gaussian model for the (regular and random) ferromagnets and the random field systems may be easily treated, and we get the expected (finite-size) scaling behavior at the critical point. The analysis of the Gaussian model for the spin glasses may not be straightforward, but I believe that the expected scaling behavior is true.

[4] For the spin glasses this is true only when we consider the model with vanishing external field.[2]

[5] For other rigorous critical exponent inequalities for the random spin systems, see ref. 3.

[6] Throughout the present paper $|x|$ means $\max\{|x_1|,..., |x_d|\} = \|x\|_\infty$.

expectation in the lattice $\Lambda(L, x)$, where each spin $\mathbf{S}_y$ in the boundary $\partial\Lambda(L, x)$ (i.e., a set of sites y satisfying $|y-x| = L$) is fixed by specific boundary conditions "b.c."

Let x be an arbitrary site with $|x| = 2L$. We consider arbitrary functions $F(\{\mathbf{S}_y\}_{y \in \Lambda(L,0)})$ and $G(\{\mathbf{S}_z\}_{z \in \Lambda(L,x)})$, which depend only on the spin variables in $\Lambda(L, 0)$ and $\Lambda(L, x)$, respectively. (0 denotes the origin of Z^d.) Then the following inequality is a simple consequence of the Markov property:

$$|\langle F(\{\mathbf{S}_y\})\, G(\{\mathbf{S}_z\})\rangle| \leqslant \max_{\text{b.c.}} |\langle F(\{\mathbf{S}_y\})\rangle_{\Lambda(L,0),\text{b.c.}}| \max_{\text{b.c.}} |\langle G(\{\mathbf{S}_z\})\rangle_{\Lambda(L,x),\text{b.c.}}| \tag{3}$$

In each term in the right-hand side, we take the maximum over all the possible boundary conditions.

In most of our applications we set $F = S_0^{(1)}$ and $G = S_x^{(1)}$, i.e., the first component of the spin variables at sites 0 and x. Then inequality (3) simply becomes

$$|\langle S_0^{(1)} S_x^{(1)}\rangle| \leqslant M_0(L, \beta)\, M_x(L, \beta) \tag{4}$$

where $|x| = 2L$ and the *finite-size order parameter* $M_x(L, \beta)$ is defined by

$$M_x(L, \beta) = \max_{\text{b.c.}} |\langle S_x^{(1)}\rangle_{\Lambda(L,x),\text{b.c.}}| \tag{5}$$

Proof of Inequality (3). We prove the inequality for a sufficiently large finite lattice Λ, and then the case for the infinite lattice follows automatically. We decompose Λ into a disjoint union as $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, where $\Lambda_1 = \Lambda(L, 0)$ and $\Lambda_2 = \Lambda(L, x)$. We also write $H = H_1 + H_2 + H_3$ where H_i for $i = 1, 2$ (not 3) are defined by restricting the first sum in (2) to the pairs $\langle x, y\rangle$, where $x \in \Lambda_i$ or $y \in \Lambda_i$, and the second sum to the sites x in Λ_i. From the definition we have

$$Z_\Lambda |\langle FG\rangle_\Lambda| \leqslant \int \prod_{y \in \Lambda_3} d\nu(\mathbf{S}_y)\, e^{-\beta H_3} \left| \int \prod_{y \in \Lambda_1} d\nu(\mathbf{S}_y)\, F e^{-\beta H_1} \right| \times \left| \int \prod_{y \in \Lambda_2} d\nu(\mathbf{S}_y)\, G e^{-\beta H_2} \right|$$

Note that for an arbitrary $\{S_z\}$ for $z \in \Lambda_3$, we have

$$\frac{|\int \prod d\nu(\mathbf{S}_y)\, F e^{-\beta H_1}|}{\int \prod d\nu(\mathbf{S}_y)\, e^{-\beta H_1}} \leqslant \max_{\text{b.c.}} |\langle F\rangle_{\Lambda(L,0),\text{b.c.}}|$$

because the left-hand side is nothing but the expectation value $|\langle F\rangle_{\Lambda(L,0),\text{b.c.}}|$, where the boundary conditions b.c. are determined by the values of $\mathbf{S}_z$ for $z \in \Lambda_3$. Substituting this bound (and the corresponding bound for Λ_2) into the first inequality, we get

$$\begin{aligned} Z_\Lambda |\langle FG\rangle_\Lambda| &\leqslant \max|\langle F\rangle| \max|\langle G\rangle| \int \prod_{y\in\Lambda_3} d\nu(\mathbf{S}_y)\, e^{-\beta H_3} \\ &\quad\times \left(\int \prod_{y\in\Lambda_1} d\nu(\mathbf{S}_y)\, e^{-\beta H_1} \int \prod_{y\in\Lambda_2} d\nu(\mathbf{S}_y)\, e^{-\beta H_2}\right) \\ &= \max|\langle F\rangle| \max|\langle G\rangle|\, Z_\Lambda \end{aligned}$$

which is nothing but the desired inequality. ∎

3. APPLICATIONS

In this section I discuss some of the applications of the inequalities (3) and (4) for specific systems.

3.1. Regular Ferromagnets (with Vanishing Magnetic Field)

The model is obtained by setting $J_{xy} = J > 0$ for all $\langle x, y\rangle$ and $\mathbf{H}_x = \mathbf{0}$ for all x. In sufficiently large dimensions such a model undergoes the usual ferromagnetic phase transition. Here, inequality (4) becomes

$$\langle S_0^{(1)} S_x^{(1)}\rangle \leqslant m(L, \beta)^2, \qquad |x| = 2L$$

where $m(L, \beta) = M_0(L, \beta) = M_x(L, \beta)$ is the finite-size ferromagnetic order parameter (or magnetization). At the ferromagnetic critical point $\beta = \beta_c$, the above quantities are expected to show the power law behavior $\langle S_0^{(1)} S_x^{(1)}\rangle \approx L^{-(d-2+\eta)}$ and $m(L, \beta_c) \approx L^{-\tilde{\beta}/\tilde{\nu}}$. (Note that this relation only defines the ratio $\tilde{\beta}/\tilde{\nu}$. This notation is motivated by the finite-size scaling theory,[4] which predicts $\tilde{\beta}/\tilde{\nu} = \beta/\nu$, where β and ν are the standard critical exponents. In what follows we use similar notations to distinguish the finite-size critical exponents.) Assuming the existence of critical exponents (one can weaken this assumption as in ref. 1), we immediately get a critical exponent inequality

$$d - 2 + \eta \geqslant 2\tilde{\beta}/\tilde{\nu}$$

This inequality should be classified as a "hyperscaling inequality," since it saturates under the so-called hyperscaling hypothesis. Usually a hyperscaling inequality provides us with information about the upper critical dimension as follows.[1] If we substitute the mean field values (see footnote 3) $\eta = 0$ and $\tilde{\beta}/\tilde{\nu} = 1$ (since $\beta = 1/2$ and $\nu = 1/2$) into the inequality

$d-2+\eta \geqslant 2\tilde{\beta}/\tilde{\nu}$, we get the bound $d \geqslant 4$. This implies that complete mean-field-type critical phenomena are impossible in dimensions less than four. In terms of the upper critical dimension d_u, this implies a rigorous lower bound $d_u \geqslant 4$ for an arbitrary N-component ferromagnet. The bound is consistent with the general belief $d_u = 4$ (see, e.g., ref. 5).

The first critical exponent inequality which implies the bound $d_u \geqslant 4$ was proved by Fisher[6] for a class of single-component ferromagnets. The inequality $d-2+\eta \geqslant 2\beta/\nu'$ (with the exponents β and ν' defined in the standard ways) was first proved for a class of single-component ferromagnets[7] by a completely different method. (See also ref. 1.)

3.2. Random Ferromagnets

The model is obtained by setting $\mathbf{H}_x = \mathbf{0}$ and regarding each J_{xy} as an independent random variable described by an identical probability distribution $d\rho(J)$. Here we consider the case when J_{xy} is mostly positive, and thus a ferromagnetic phase transition takes place.[8] The inequality (4) may be averaged over the J distribution and yields

$$\overline{\langle S_0^{(1)} S_x^{(1)} \rangle} \leqslant \overline{|\langle S_0^{(1)} S_x^{(1)} \rangle|} \leqslant m(L, \beta)^2, \qquad |x| = 2L$$

where $m(L, \beta) = \overline{M_0(L, \beta)} = \overline{M_x(L, \beta)}$ is the finite-size magnetization. This bound at $\beta = \beta_c$ again leads us to a hyperscaling inequality

$$d-2+\eta \geqslant 2\tilde{\beta}/\tilde{\nu}$$

where the exponents are defined as in Section 3.1. (It is not difficult to extend the argument in ref. 1 to prove $d-2+\eta \geqslant 2\beta/\nu'$, where β and ν' are defined in the standard way.) Again this implies the lower bound $d_u \geqslant 4$ because the mean field theory for the random ferromagnets is exactly the same as that for the regular ferromagnets (see footnote 3). The result is consistent with the Harris criterion,[9] which indicates that the critical phenomena in the random ferromagnets differ from those in the regular ferromagnets only when $\alpha > 0$.

Note that the basic inequality (4) is valid for unaveraged correlation functions as well. However, the averaging procedure seems to be necessary to get a meaningful consequence, since the quantity $M_x(L, \beta)$ crucially depends on the sample.

3.3. Spin Glasses

The model is formally the same as Section 3.2, but now the probability distribution $d\rho(J)$ is assumed to be invariant under the transformation

$J \to -J$. {Typically we set $d\rho(J) = [\frac{1}{2}\delta(J - J_0) + \frac{1}{2}\delta(J + J_0)]\, dJ$ or $d\rho(J) = \text{const} \cdot \exp(-J^2/2J_0^2)\, dJ$.} In sufficiently large dimensions such a model is expected to undergo the so-called spin glass transition (e.g., ref. 10). In the spin glass transition, the relevant order parameter is the Edwards–Anderson order parameter $q = \overline{\langle S_0^{(1)} \rangle^2}$, whose finite-size counterpart should be defined as

$$q(L, \beta) = \overline{\max_{\text{b.c.}} (|\langle S_0^{(1)} \rangle_{\Lambda(L,0),\text{b.c.}}|)^2} = \overline{M_0(L, \beta)^2} = \overline{M_x(L, \beta)^2}$$

Now by squaring both sides of the inequality (4) and taking the J average, we get

$$\overline{\langle S_0^{(1)} S_x^{(1)} \rangle^2} \leqslant q(L, \beta)^2. \qquad |x| = 2L$$

At the spin glass critical point β_{SG} these quantities are expected to behave as

$$\overline{\langle S_0^{(1)} S_x^{(1)} \rangle^2} \approx L^{-(d-2+\eta_{\text{EA}})}, \qquad q(L, \beta_{\text{SG}}) \approx L^{-\tilde{\beta}_{\text{EA}}/\tilde{v}}$$

Thus we get a hyperscaling inequality

$$d - 2 + \eta_{\text{EA}} \geqslant 2\tilde{\beta}_{\text{EA}}/\tilde{v}$$

which yields a lower bound $d_u \geqslant 6$ when we substitute the mean field values (see footnotes 3 and 4)[2,11] $\eta_{\text{EA}} = 0$ and $\tilde{\beta}_{\text{EA}}/\tilde{v} = 2$ (since $\beta_{\text{EA}} = 1$ and $v = 1/2$). This bound is consistent with the general belief $d_u = 6$ for the upper critical dimension of the spin glasses.

3.4. Random Field Systems

Here we set $J_{xy} = J > 0$ as in Section 3.1, but regard each $\mathbf{H}_x$ as an independent random variable described by an identical probability distribution $d\rho(\mathbf{H})$. We assume that $d\rho(\mathbf{H})$ is invariant under the transformation $\mathbf{H} \to -\mathbf{H}$. {Typically we set $d\rho(\mathbf{H}) = [\frac{1}{2}\delta(\mathbf{H} - \mathbf{H}_0) + \frac{1}{2}\delta(\mathbf{H} + \mathbf{H}_0)]\, d^N\mathbf{H}$ or $d\rho(\mathbf{H}) = \text{const} \cdot \exp(-\mathbf{H}^2/2H_0)\, d^N\mathbf{H}$.} In sufficiently large dimensions, this model undergoes a ferromagnetic transition[12] (when $\overline{\mathbf{H}}^2$ is not too large). Because the present Hamiltonian is typically not invariant under $\mathbf{S}_x \to -\mathbf{S}_x$, the quantity $M_x(L, \beta)$ defined as in (5) does not vanish in the $L \to \infty$ limit at any temperature. However, an appropriate finite-size order parameter is obtained by also executing a spatial average as follows:

$$m(L, \beta) = \overline{\max_{\text{b.c.}} \left| \left\langle L^{-d} \sum_{y;\, |y| \leqslant L/2} S_y^{(1)} \right\rangle_{\Lambda(L,0),\text{b.c.}} \right|}$$

We use our basic inequality (3) by setting $F = L^{-d} \sum_{y;\, |y| \leqslant L/2} S_y^{(1)}$ and $G = L^{-d} \sum_{z;\, |z-x| \leqslant L/2} S_z^{(1)}$. After taking the H average, we get

$$L^{-2d} \sum_{\substack{y, z;\, |y| \leqslant L/2 \\ |z-x| \leqslant L/2}} \overline{\langle S_y^{(1)} S_z^{(1)} \rangle} \leqslant m(L, \beta)^2, \qquad |x| = 2L$$

At the critical point $\beta = \beta_c$, the two-point function is expected[7] to decay as $\overline{\langle S_y^{(1)} S_z^{(1)} \rangle} \approx |y-z|^{-(d-4+\bar{\eta})}$. Thus, the left-hand side should decay as $L^{-(d-4+\bar{\eta})}$. We also expect $m(L, \beta_c) \approx L^{-\min(\tilde{\beta}/\tilde{\nu}, d/2)}$, so we get a hyperscaling inequality

$$d - 4 + \bar{\eta} \geqslant 2 \min(\tilde{\beta}/\tilde{\nu}, d/2)$$

The mean-field values (see footnote 3) $\bar{\eta} = 0$ and $\tilde{\beta}/\tilde{\nu} = 1$ (since $\beta = 1/2$ and $\nu = 1/2$) yield a lower bound $d_u \geqslant 6$ which is consistent with the result of the dimensional reduction[13] $d_u = 6$. Although the dimensional reduction predicts the wrong lower critical dimension (at least for the Ising model), it is believed that it predicts the correct upper critical dimension.

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[7] Note that the truncated two-point function at $\beta = \beta_c$ is believed to behave as $\overline{\langle S_0^{(1)} S_x^{(1)} \rangle - \langle S_0^{(1)} S_x^{(1)} \rangle} \approx L^{-(d-2+\eta)}$, where the identity $\bar{\eta} = 2\eta$ is conjectured.[13]

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